The Polarization of Economics

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Abstract

The focus of the financial instability hypothesis of Hyman Minsky (1992) is on an accumulating capitalist economy that moves through real calendar time. A couple of years ago, the author developed an economic modeling, called dynamic analysis, to study this accumulation process by means of computer simulations (Kremer, 2008, 2013).

For the special case of constant parameters, which is presented in this paper, we give closed formulas for the wealth and other macroeconomic quantities of both the total economy and individual household groups as functions of time. In this case of a so-called constant parameter economy we analyze mathematically, under which conditions an economy eventually becomes polarized, i.e., when the distribution of wealth turns out to become extremely unequal. It will be shown that polarization can be avoided only if either capital income is bounded, which translates to $r = 0$ in the constant parameter model we discuss here, or if there is eternal economic growth above the capital income rate, and this translates to $y > r$. But in the latter case additional requirements have to be fulfilled to prevent the economy from becoming polarized.

The results are in accordance with the research findings of Thomas Piketty (2014, 2015) and the model presented substantiates Minsky's instability hypothesis as well as serving as a theoretical basis for an explanation of Piketty's empirical results.

Keywords: Financial Instability Hypothesis, Polarization, Thomas Piketty.

JEL: B12, B15, B16

Introduction

In his 1992 paper, Hyman Minsky explains his *Financial Instability Hypothesis* and rejects explicitly neoclassical economics:

> These historical episodes are evidence supporting the view that the economy does not always conform to the classic precepts of Smith and Walras: they implied that the economy can best be understood by assuming that it is constantly an equilibrium seeking and sustaining system (Minsky, 1992).

Minsky's theory relies on Keynes's General Theory as well as on the credit view of money and finance proposed by Joseph Schumpeter. Minsky explicates:

> The theoretical argument of the financial instability hypothesis starts from the characterization of the economy as a capitalist economy with expensive capital assets and a complex, sophisticated financial system. The economic problem is identified following Keynes as the "capital development of the economy," rather than the Knightian "allocation of given resources among alternative employments." The focus is on an accumulating capitalist economy that moves through real calendar time (Minsky, 1992).

In contrast to neoclassical economics, Minsky identifies financial markets as institutions exercising great impact on the development of economies. But, in an amendment to Keynes and Minsky, in recent times it has become more and more evident that not only "In a more complex (though still highly abstract) structure, aggregate profits equal aggregate investment plus the government deficit (ibid)", but that in our economies approximately all existing money is created by credit. Central banks or governments do not provide money as supply for the economies, but commercial banks create money as entries in their balance sheets via credit contracts. Therefore, interest has to be paid to commercial banks for nearly all existing money circulating in the economies. For a clear explanation of the structure of our monetary systems and for suggestions of how they might be reformed (see Jackson and Dryson, 2012; Ryan-Collins et al., 2012).

Minsky presents key aspects of his *Financial Instability Hypothesis* as follows:

Three distinct income-debt relations for economic units, which are labeled as hedge, speculative, and Ponzi finance, can be identified. ... In contrast, the greater the weight of speculative and Ponzi finance, the greater the likelihood that the economy is a deviation amplifying system. ... In particular, over a protracted period of good times, capitalist economies tend to move from a financial structure dominated by hedge finance units to a structure in which there is large weight to units engaged in speculative and Ponzi finance (Minsky, 1992).

Minsky did not formulate his hypothesis as a mathematical model. This was worked out by Steve Keen:

Minsky' s own attempts to devise a mathematical model of his hypothesis were unsuccessful, arguably because the foundation he used – the multiplier-accelerator model – was itself flawed (Keen, 2000, pp.84– 89). Keen (1995) instead used Goodwin' s growth cycle model (Goodwin, 1967), which generates a trade cycle with growth out of a simple deterministic structural model of the economy (Keen, 2011).

Keen developed for the first time an economic model (see Keen, 2010, 2012) that showed a debt-induced breakdown of the model economy as one of its possible states.

In the present paper, "the focus is on an accumulating capitalist economy that moves through real calendar time." It seems obvious that concentration of wealth combined with a – natural – decrease of economic growth increases the danger of speculative and Ponzi finance.

The concentration of capital is empirically well verified and was recently thoroughly investigated and amply documented by Thomas Piketty in his *Capital in the Twenty-First Century*. We will study the fundamental self-sustaining disequilibrating processes that lead to the accumulation of capital eventually possessed by a wealthy minority that becomes powerful enough to drive the economy into one of its breakdown states.

The Cycle Model

In this section we present the cycle model from which dynamic analysis is derived. In Figure 1 the nodes *households* and *firms* represent the corresponding economic agents. We analyze the flows in and out of each node, equate them, and obtain accounting equations that constitute the cornerstones of the model.

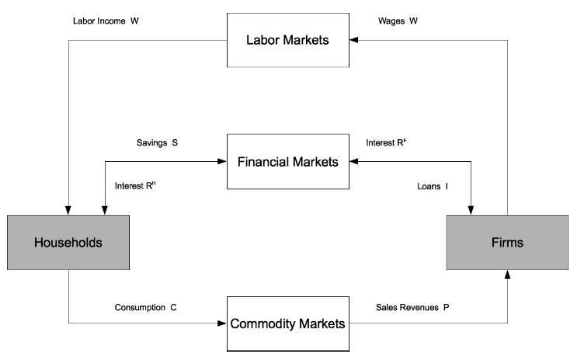

Figure 1: Economic cycle

The Households

Households have two sources of income, wages W and interest income R^H. W originates from labor and is therefore also called labor income. In contrast, R^H is capital income and stems from the ownership of wealth V. Income $W + R^H$ of the households is divided between consumption expenditures C and savings S, thus

$$W + R^H = C + S. \tag{2.1}$$

If S is positive, then V is increased by the aggregated amount of savings. If S were negative, however, then the saved wealth V of the households would decrease by S. Even if S is positive, some households may have negative savings, corresponding to a reduction of their wealth or to a loan.

The Firms

Income of firms consists of sales revenues P and of loans I that have been taken out. Expenditures are comprised of wages W and some part of the financing costs R^F. With R^F we denote that share of the financing costs that the banks hand over to their depositors. The remaining difference is part of the revenues of the banks and is therefore an integral part of aggregated income W. Thus we have

$$P + I = W + R^F. \tag{2.2}$$

Macroeconomics Relationship Between Households and Firms

From the economic cycle we read off:

$$P = C, \tag{2.3}$$

the sales revenues of the firms correspond to the consumption expenditures of the households.

In our financial systems, increase of aggregated balances is realized via credit expansion, and every decrease of aggregated balances is realized by amortization or write-off. This translates to:

$$S = I. \tag{2.4}$$

Equation (2.4) states that every change of aggregated monetary wealth by an amount S is mirrored by the same quantity I of aggregated debt. In particular aggregated monetary wealth corresponds to aggregated debt B in the economy.

Thus, if savings S occurs in the economy, then some part of the wages paid by the firms is not used for consumption C, and the firms have a demand for credit I to the amount of savings S. Alternatively, investments might be undertaken in the economy. In this case firms take out loans I, and the corresponding deposits S will be transferred to the accounts of the borrowers – and hereafter to other accounts of members of the economy.[23]

Further, we have:

$$R^H = R^F =: R. \tag{2.5}$$

The amount R^F is the fraction of the financing costs that is handed over to the depositors as interest revenues R^H. (2.5) does not imply that interest rates for credits and savings coincide. The remaining fraction of the financing costs paid by the borrowers may be interpreted as labor income of the banks and is thus part of W. R^H and R^F coincide by definition and are thus abbreviated by the common symbol R. From (2.1), (2.2), (2.3), (2.4) and (2.5) we obtain

$$\begin{array}{ccccccc} W & + & R^H & = & C & + & S \\ \parallel & & \parallel & & \parallel & & \parallel \\ W & + & R^F & = & P & + & I \end{array} \tag{2.6}$$

Now, gross domestic product *(gdp)* Y is defined as

$$Y = W + R = P + I = C + S. \tag{2.7}$$

This quantity may optionally be interpreted as the sum of all expenditures of the firms $W + R$, as the sum of all revenues by the firms $P + I$, as the sum of all revenues of the households $W + R$ or as the sum of all expenditures of the households $C + S$. Economic growth is defined as growth of gross domestic product Y.

In particular, the representation :

$$Y = W + R \tag{2.8}$$

is instructive. Assume there is no economic growth, but interest revenues rise due to compound interest effects. In this case aggregated labor income is forced to decrease, and an increasingly larger

[23] The rule savings = investments that is found in economic textbooks does not represent what we denote by S = I. By mistake, in the standard literature it is frequently assumed that we have a monetary supply system and that the savings of the depositors are lent out to borrowers, thereby realizing S = I. But this is not true for our monetary credit systems (see Kremer 2013, 2016).

fraction of gross domestic product will be distributed among the capital owners of the economy. But if it is possible to generate sufficient economic growth, then labor wages W may grow or at least remain stable, even if interest revenues R increase. Thus, the need to limit the fraction of interest revenues in gdp forces the economy to grow.

The Influence of the Monetary System on the Model

Bank notes, coins and the balances on current accounts are vouchers. Monetary systems are voucher systems. The advantage of voucher systems is that the exchange of goods and services is decoupled and simplified. A consequence of a voucher is that it usually has no intrinsic value; a voucher is worth what you can buy for it. Vouchers do not even need to be composed of matter, numbers on current accounts or on cash cards suffice.

There are several conceivable voucher systems:

• In monetary supply systems, vouchers are provided by governmental institutions such as central banks. Here the aggregated money supply is controlled by the issuing institution and banks are intermediaries that collect savings amounts from the depositors and lend vouchers to borrowers. The borrowers are obligated to pay interest to the depositors who make savings amounts available.

• In a monetary credit system, vouchers are created by credit and destroyed by amortization. In a monetary credit system, vouchers may be created by private banks, and the central bank is only needed for providing the economy with state-approved vouchers that commercial banks borrow from the central bank to the extent that is needed for day-to-day business. Because money is created via credit, private banks earn interest income for each voucher circulating in the economy.

•In a positive money system, the two previous systems are combined. The system might consist of a base supply system that is controlled by a governmental but hopefully independent institution. The author proposes the combination of this supply system with a monetary credit system. If a borrower takes out a loan from a commercial bank, it is suggested that this bank is obligated to borrow the credit amount completely from the central bank and hand it over to the borrower. In the course of redemption, the money flows back to the central bank. Commercial banks should be allowed to charge service revenues and risk premiums, but there is no need for the payment of interest. The amount of credit could be controlled by the central bank via specification of amortization amounts and redemption times.

The economic cycle represented in Figure 1 shows that, in the aggregate, the firms provide the supply of money I and that this money is distributed as deposits S among the households.

The flows in the economic cycle occur periodically each year. Therefore we will label each quantity with a lower time index. If we denote aggregated monetary wealth at the beginning of some initial year 0 with V_0, then aggregated wealth at the end of year t is

$$V_t = V_0 + S_1 + \cdots + S_t = V_0 + \sum_{i=1}^{t} S_i.$$

Aggregated monetary wealth V_t may be interpreted as sum of the deposit accounts of all households in the economy plus cash. The node *financial markets* allows for changes of Y; thus, this node may act as a source or sink in the cycle model and depends on the financial system as well as on political and investment decisions.

Now we investigate the impact of the financial system on the economy:

• In monetary supply systems no additional savings occur if the total amount of money is held constant. Loans are financed with savings, so that $S_i = 0$ and $V_t = V_0$. Interest income R^H is received by those households that have lent out money. Nevertheless, $S_t \neq 0$ will occur if the central bank decides to change the money supply. For example, the central bank could make additional money available to the government, and this money could subsequently be distributed among the members of the economy within the scope of government spending.

• In monetary credit systems the total amount of money is not constant, in particular $S_i > 0$ for all i is possible. Since money is solely created by credit, the aggregated amount of money is mirrored by the aggregated amount of debt, and total debt is subject to interest payments.

• In a positive money system a credit system is superimposed on a supply system. In the version we suggest here, credits are provided by commercial banks, but commercial banks are obligated to borrow each credit amount completely from the central bank, so that credits do not require savings and in particular there is no need for interest payments and interest revenues.

Thus, the monetary system has a strong impact on capital revenues. In a positive money system, money is not created via credit and owners of monetary wealth do not receive interest income. In a monetary supply system, money is not created via credit and only those owners of monetary wealth who lend money receive interest income. In a monetary credit system all money is created via credit and total debt is subject to interest payments. On the other hand, in monetary credit systems savings are not used for credits and therefore need not be rewarded by interest payments, but they are in our current monetary systems. Thus, monetary credit systems are the most profitable alternative for commercial banks and wealthy customers, and these are just the systems we actually have.

Consideration of Real Assets

In addition to the acquisition of real assets for consumption, households can also invest in profitable real capital. Thus, properties and real estate can be purchased and subsequently be rented or leased out. Moreover, money can be invested in stocks to acquire corporate shares. If the corresponding companies pay dividends, the investors participate in company profits. In any case, the revenues of rents, leases and dividends are acquired due to the ownership of capital, analogous to the revenue of interest income due to the ownership of monetary wealth.

Figure 2 shows the extension of the cycle model to include investments in real assets. S^M denotes monetary savings, and S^K savings in real assets. Capital income due to the ownership of real assets is denoted with R^K.

If households buy newly-issued stocks, the purchase prices S^K lead to a cash inflow I^K for the corresponding company. If dividends are paid, then these are to be interpreted as capital revenues R^K. But if a household buys stocks from another household, then S^K and S^M remain unchanged, and the households involved exchange shares of S^M and S^K.

Households that rent or lease properties or real estate establish a firm, and the corresponding capital savings S^K flow into these firms as investments I^K. The households that make use of these properties and real estate pay rents and leases as part of their consumption expenditures C to the providing firms.

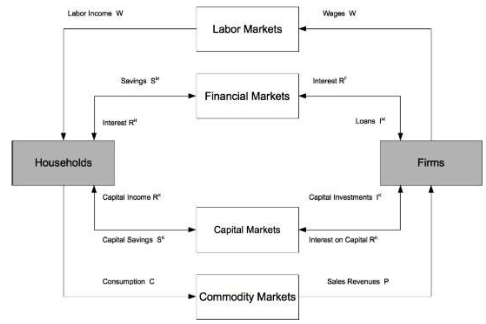

Figure 2: Economic cycle with financial and tangible assets

Thus, the balance equation of the households has to be extended to:

$$W + R^H + R^K = C + S^M + S^K, \tag{2.9}$$

and for the companies we obtain:

$$W + R^F + R^K = P + I^M + I^K. \tag{2.10}$$

We still have:

$$C = P,$$

$$R^H = R^F,$$

$$S^M = I^M$$

as well as:

$$S^K = I^K.$$

With the definitions:

$$S = S^M + S^K,$$

$$I = I^M + I^K,$$

$$R = R^H + R^K$$

we are led to:

$$S = I$$

$$W + R = C + S = P + I =: Y,$$

and this coincides with (2.7). But now, the quantity R denotes capital revenues resulting from ownership of monetary wealth and of real assets.

However, the statement that aggregated wealth coincides with aggregated debt is valid only for financial assets. If we denote monetary wealth with V^M, tangible assets with V^K and total wealth with V, we obtain:

$$V_t^M = V_{t-1}^M + S_{t-1}^M$$

$$V_t^K = V_{t-1}^K + S_{t-1}^K$$

$$V_t = V_t^M + V_t^K,$$

But:

$$V_t^M = B_t,$$

where B denotes aggregated debt.

Dynamic Analysis

We model N classes of households which differ in their wealth, consumption, savings, and income. To do this we think of the economy as being divided into N groups of equal numbers of households with equal macroeconomic properties. We use upper group indices to denote the values of macroeconomic quantities for household groups. So, W^i denotes the labor income of the ith group, $i = 1, \dots, N$. The corresponding symbol without upper index denotes the aggregated value, so that $W = \sum_{i=1}^{N} W^i$.

Now we consider the annual development of an economy over a period from today, $t = 0$, until T years in the future. We characterize all the quantities that appear through a lower time index t. Thus, we denote W_t^i to be the labor income of group i in year t and write $W_t = \sum_{i=1}^{N} W_t^i$.

We assume that for each group $i = 1, \dots, N$ some initial wealth V_{-1}^i is specified for year $t = -1$.[24] With r_t we denote the averaged annual interest rate for the time interval from $t-1$ to t. We model economic growth with growth factors y_t for the years $t = 1, \dots, T$ by

$$Y_t = (1 + y_t)Y_{t-1},$$

where Y_t denotes the gross domestic product of year t. We use the representations

$$Y_t = W_t + R_t = C_t + S_t,$$

[24] For the quantity wealth alone we start with year -1, so that $R_0 = rV_{-1}$ that is interest income for initial year 0 can be formulated.

where $R_t = r_t V_{t-1}$ denotes total interest payments on wealth, V_{t-1} aggregated wealth in year $t-1$, C_t total consumption, and S_t total savings of the economy. Thus,

$$W_t = Y_t - r_t V_{t-1}.$$

W_t is to be distributed among the household groups. For $t = 0$ we specify an initial income distribution $W_0^1, \dots, W_0^N$, and for later years $t > 0$ we define factors $\alpha_t^i \geq 0$ based on rules or on empirical data with $\sum_{i=1}^N \alpha_t^i = 1$ so that:

$$W_t^i = \alpha_t^i W_t.$$

Thus, for each year t we have $\sum_{i=1}^N W_t^i = \left(\sum_{i=1}^N \alpha_t^i\right) W_t = W_t$. We further assume that overall consumption changes with certain growth rates c_t, $t = 1, \dots, T$, so that we have:

$$C_t = (1 + c_t) C_{t-1}.$$

Moreover, individual household groups have to contribute to overall consumption C_t for $t = 1, \dots, T$. For this we define, analogous to wages, an initial distribution $C_0^1, \dots, C_0^N$ of consumption and for $t > 0$ factors $\beta_t^i > 0$ with $\sum_{i=1}^N \beta_t^i = 1$, so that:

$$C_t^i = \beta_t^i C_t$$

and $\sum_{i=1}^N C_t^i = C_t$. From this the amount of savings per household group is calculated,

$$S_t^i = W_t^i + R_t^i - C_t^i.$$

Finally the amounts of wealth in year t work out as:

$$V_t^i = V_{t-1}^i + S_t^i.$$

This procedure is now iterated over time.

Besides the presetting of initial data, the algorithm requires the specification of

1. the growth rates c_t of total consumption C_t,
2. the distribution factors β_t^i for total consumption C_t,
3. and the distribution factors α_t^i for total wages W_t.

Once these data and rules are specified, the algorithm is determined completely. One possibility for achieving this consists in computing the data listed above from historical data and to extrapolate them in some appropriate way into the future. Alternatively, rules may be specified for these data. In this paper we choose the growth rates and the distribution factors to be constant over time to obtain what we call a constant parameter economy.

Interest Transfer

If we divide interest into percentages of consumption among households, we obtain as the interest payment hidden in consumption Π_t^i of the ith household group the quantity:

$$\Pi_t^i = \frac{C_t^i}{C_t} R_t = \beta_t^i R_t.$$

We define the balance Z_t^i of interest income R_t^i and interest payments Π_t^i of the ith household group as:

$$Z_t^i = R_t^i - \Pi_t^i. \tag{4.1}$$

We call Z_t^i the *interest transfer* of group i at time t. Obviously, we have:

$$\sum_{i=1}^{N} Z_t^i = \sum_{i=1}^{N} R_t^i - \frac{R_t}{C_t}\sum_{i=1}^{N} C_t^i = 0.$$

To demonstrate the effect of interest transfer we consider the example of an economy consisting of three household groups to which we assign the following initial data:

Group	Income	Consumption	Wealth
1	10	10	0
2	50	30	100
3	100	50	500

Table 1: Initial data

The three groups represent the lower class, the middle class and the upper class of the economy, respectively. The lower class has no wealth and comparatively low income that is consumed completely. The second group owns some wealth, has more income and consumes more than the lower class, but not all. The upper class has even more wealth, income and expenditures, but consumes only a fraction of 50% of its income.

The total wealth of the economy, which for the moment we assume to be purely monetary, is $V = 0 + 100 + 500 = 600$. If we assume the interest rate to be $r = 3\%$, we find total interest income of the economy to be $R = r \cdot V = 3\% \cdot 600 = 18$. Because money is created via credit, total debt of the economy and the financing costs also amount to 600 and 18, respectively. Here we do not take the state as economic agent into account, and therefore the firms are indebted in total with 600 and bound to pay the financing costs of 18. But the financing costs are part of the total costs of the products and services manufactured and thus are integrated into the prices that will be paid by the households in the context of their consumption expenditures.

In our example, for the total consumption of the economy we find $C = 10 + 30 + 50 = 90$, and thus we calculate its interest payments hidden in consumption $\Pi^i = \frac{C^i}{C} R$ for each household group:

Group	Share in Consumption	Interest Payment in Consumption Π^i
1	$\frac{10}{90} = 11.1\%$	$18 \times \frac{1}{9} = 2$
2	$\frac{30}{90} = 33.33\%$	$18 \times \frac{3}{9} = 6$
3	$\frac{50}{90} = 55.55\%$	$18 \times \frac{5}{9} = 10$

Table 2: Allocation of interest payment in consumption to each household group

Next we calculate the interest income of each household group $R^i = rV^i$:

Group	Wealth	Interest Income R^i
1	0	$0 \times 3\% = 0$
2	100	$100 \times 3\% = 3$
3	500	$500 \times 3\% = 15$

Table 3: Interest Income per Household Group

Now the interest transfer $Z^i = R^i - \Pi^i$ for each household group is given as the difference between its corresponding interest income and its interest payment via consumption:

Group	$Z^i = R^i = \Pi^i$
1	$0 - 2 = -2$
2	$3 - 6 = -3$
3	$15 - 10 = 5$

Table 4: Interest Transfer per Household Group

We see that the upper class has a positive balance and that its capital income is paid by the other household groups. This transfer mechanism has far-reaching consequences; it is one of the causes – perhaps the principal cause – for the redistribution of wealth and will be explored further below.

If we allow wealth V to consist of monetary and tangible assets, then we have to consider the capital income R that is generated from V, and we have to interpret r as the average capital return rate, that is $r = R/V$. But capital income R, e.g. interest on financial assets, rents, leases and dividends, has to be paid as part of the consumption expenditures of the households, and again we are led to the above formulas for the interest transfer.

Closed Solution of Dynamic Analysis for a Constant Parameter Economy

We consider dynamic analysis without state for the special case of constant capital return rates $r_t = r > -1$, and we assume constant economic growth $y_t = y > -1$, so that:

$$Y_t = (1+y)^t Y_0. \tag{5.1}$$

Let $S_0 \geq 0$ and $C_0 > 0$. Further we assume that total consumption increases with a constant growth factor c, so that:

$$C_t = (1+c)^t C_0.$$

Thus we find:

$$S_t = Y_t - C_t = Y_0(1+y)^t - C_0(1+c)^t,$$

and this leads to an asymptotic evolution of savings rates S_t/Y_t given by:

$$\frac{S_t}{Y_t} = 1 - \left(\frac{1+c}{1+y}\right)^t \frac{C_0}{Y_0} \to \begin{cases} 1 & if \quad c < y \\ S_0/Y_0 & if \quad c = y \\ -\infty & if \quad c > y \end{cases}$$

for $t \to \infty$. Thus, modeling of realistic saving rates implies the assumption that total consumption grows with gross domestic product, so we set $c = y$ and:

$$C_t = (1+y)^t C_0. \tag{5.2}$$

Further we presuppose time independent shares of total income W_t and of total consumption C_t for each group,

$$W_t^i = \alpha^i W_t, \quad C_t^i = \beta^i C_t, \tag{5.3}$$

with constant factors $0 \leq \alpha^i \leq 1$, $\sum_{i=1}^N \alpha^i = 1$, and $0 \leq \beta^i \leq 1$, $\sum_{i=1}^N \beta^i = 1$. An economy with these constant parameter properties is called a *constant parameter economy* (without state).

Evolution of the Economy

Proposition 1. *For aggregated savings of the households S_t and for the total wealth V_t of a constant parameter economy we have:*

$$S_t = Y_t - C_t = (1+y)^t S_0 \tag{5.4}$$

And:

$$V_t = \begin{cases} (t+1)S_0 + V_{-1} & if \quad y = 0 \\ \dfrac{(1+y)^{t+1}}{y} S_0 + \left(V_{-1} - \dfrac{S_0}{y}\right) & if \quad y \neq 0. \end{cases} \tag{5.5}$$

Proof. (5.4) follows from (5.1) and (5.2). (5.4) and $V_t = V_{t-1} + S_t$ inductively leads to

$$V_t = V_{-1} + S_0 \sum_{j=0}^{t} (1+y)^j$$
$$= \begin{cases} V_{-1} + (t+1)S_0 & if \quad y = 0 \\ V_{-1} + \dfrac{(1+y)^{t+1} - 1}{y} S_0 & if \quad y \neq 0, \end{cases}$$

and the assertion follows.

Remark 2. *Our general prerequisite on interest and growth rates is that they are both assumed to be* > -1. *Thus, here and henceforth* $y \neq 0$ *has to be interpreted as* $y > -1$ *and* $y \neq 0$; *likewise* $r \neq 0$ *is interpreted as* $r > -1$ *and* $r \neq 0$.

Remark 3. *From* $V_t = V_{-1} + S_0 \sum_{j=0}^{t} (1+y)^j$ *we conclude* $V_t \geq 0$ *for all* t, *if* $S_0 \geq 0$ *and* $V_{-1} \geq 0$.

Remark 4. *In Capital and the Twenty-First Century, Thomas Piketty formulates two equations that he calls fundamental laws of capitalism. The first is stated on p. 52 as:*

$$\alpha = r\beta,$$

where α denotes the share of income from capital in national income, β is the capital/income rate and r is the capital return rate. With the notations used in this article, we translate this to:

$$\alpha = \frac{R_t}{Y_t}, \quad \beta = \frac{V_{t-1}}{Y_t},$$

so that $\alpha = r\beta$ becomes

$$R_t = rV_{t-1},$$

and this is one of the specifications of section 3.

On p. 166, the second law is formulated as:

$$\beta = s/g$$

in the long run, where s is the savings rate and g the economic growth rate. With the notations used here this transfers to:

$$s = \frac{S_t}{Y_t}, \quad g = y,$$

thus the second law translates to:

$$\frac{V_{t-1}}{Y_t} = \frac{1}{y}\frac{S_t}{Y_t}$$

for large t. With (5.5) we find:

$$\frac{V_{t-1}}{Y_t} = \frac{M + \frac{1}{y}S_t}{Y_t} \approx \frac{1}{y}\frac{S_t}{Y_t} = \frac{1}{y}\frac{S_0}{Y_0},$$

for large t, because $M/Y_t \to 0$ for $t \to \infty$. Therefore, the second law is also in accordance with the current model.

Corollary 5. *Let $V_{-1} \geq 0$ and $S_0 > 0$, then for the relative change of total wealth $\frac{V_t - V_{t-1}}{V_{t-1}}$ we have asymptotically*

$$\lim_{t\to\infty} \frac{V_t - V_{t-1}}{V_{t-1}} = \begin{cases} y & if \quad y > 0 \\ 0 & if \quad -1 < y \leq 0. \end{cases}$$

Furthermore, $\frac{V_t - V_{t-1}}{V_{t-1}} > 0$ for t large enough.

Proof. If $y = 0$, then:

$$\frac{V_t - V_{t-1}}{V_{t-1}} = \frac{S_0}{V_{-1} + tS_0} \downarrow 0$$

for $t \to \infty$. In the case of $y \neq 0$, a short calculation leads to:

$$\frac{V_t - V_{t-1}}{V_{t-1}} = \frac{S_t}{V_{t-1}} = \frac{y}{1 + \frac{1}{(1+y)^t}\left(\frac{yV_{-1}}{S_0} - 1\right)}.$$

If $y > 0$, then $\lim_{t\to\infty} \frac{1}{(1+y)^t} = 0$, thus $\frac{V_t - V_{t-1}}{V_{t-1}} \to y$ for $t \to \infty$. If $-1 < y < 0$, then $\lim_{t\to\infty} \frac{1}{(1+y)^t} = \infty$, and therefore $1 + \frac{1}{(1+y)^t}\left(\frac{yV_{-1}}{S_0} - 1\right) \to -\infty$, thus $\frac{V_t - V_{t-1}}{V_{t-1}} \downarrow 0$, as was to be shown.

Thus, growth of total wealth depends on economic growth y, but not on the capital return rate r.

Proposition 6. *In a constant parameter economy, total wage is given by:*

$$W_t = \begin{cases} -trS_0 + W_0 & if \quad y = 0 \\ (1+y)^t B + A & if \quad y \neq 0. \end{cases} \tag{5.6}$$

where

$$A = r\left(\frac{S_0}{y} - V_{-1}\right) \tag{5.7}$$

and

$$B = Y_0 - \frac{r}{y} S_0. \tag{5.8}$$

Proof. For $y = 0$ we find with (5.1) and (5.2)

$$Y_t = Y_0, \quad C_t = C_0.$$

(5.4) and (5.5) imply $S_t = Y_0 - C_0 = S_0$ and

$$V_t = V_{-1} + (t+1)S_0.$$

Due to $R_t = rV_{t-1}$ we conclude:

$$\begin{aligned} W_t &= Y_t - rV_{t-1} \\ &= Y_0 - R_0 - trS_0 \\ &= W_0 - trS_0. \end{aligned}$$

For $y \neq 0$ we find with (5.5)

$$\begin{aligned} W_t &= Y_t - R_t \\ &= (1+y)^t Y_0 - rV_{t-1} \\ &= (1+y)^t \left(Y_0 - \frac{r}{y} S_0\right) + r\left(\frac{S_0}{y} - V_{-1}\right), \end{aligned}$$

as was to be shown.

Contrary to total wealth, the evolution of total labor income does not only depend on economic growth y, but also on the capital return rate r. Assume $V_{-1} \geq 0$, $S_0 > 0$, and $r > 0$. From proposition 6 we conclude that for $y > 0$ total labor income W_t will increase exponentially in the case of $yY_0 > rS_0$, this being equivalent to

$$yY_t > rS_t.$$

Thus, if the increments of gdp yY_t are larger than interest on the increments of total wealth rS_t, then total labor income will increase. But in the case of $yY_0 < rS_0$, total labor income will decrease to and below zero. But for $y > 0$ total wealth will increase exponentially, and if economic growth is not strong enough this already indicates bleak prospects for the economic development of household groups with low wealth.

Evolution of the Household Groups

Now we analyze individual household groups. For $i = 1, \dots, N$ we have by definition

$$S_t^i = W_t^i + R_t^i - C_t^i \tag{5.9}$$

$$= \alpha^i W_t + rV_{t-1}^i - \beta^i C_t$$

and

$$V_t^i = V_{t-1}^i + S_t^i. \tag{5.10}$$

Proposition 7. *For labor income W_t^i and consumption expenditures C_t^i of a household group $i = 1, \dots, N$ we have*

$$W_t^i = \begin{cases} -tr\alpha {S_0}^i + W_0^i & (y = 0) \\ (1+y)^t \alpha^i B + \alpha^i A & (y \neq 0), \end{cases}$$

where A and B are given by (5.7) and (5.8), and

$$C_t^i = (1+y)^t C_0^i$$

Proof. The assertions follow from proposition 6, (5.2) and

$$\alpha^i = \frac{W_t^i}{W_t}, \quad \beta^i = \frac{C_t^i}{C_t}.$$

The assumption of constant parameters in the model leads to an underestimation of polarization effects because empirical studies prove that labor income of the lower household groups has dropped during the previous decades, in contrast to increased labor income of top earners. Thus, the α^i factors of top earners have increased whereas those of the general population have decreased. But we will see that even the prerequisite of constant α^i factors leads to massive redistribution effects.

Proposition 8. *In a constant parameter economy the wealth V_t^i of the i-th household group is for $r = 0$ given by*

$$V_t^i = \begin{cases} (t+1)S_0^i + V_{-1}^i & (y=0) \\ \dfrac{\theta^{t+1}-1}{y} S_0^i + V_{-1}^i & (y \neq 0) \end{cases}$$

(5.11)

and for $r \neq 0$ *by*

$$V_t^i = \begin{cases} \frac{\rho^{t+1}-1}{r}(W_0^i - C_0^i) - \left(\sum_{j=0}^{t-1}(t-j)\rho^j\right)\alpha^i r S_0 + \rho^{t+1}V_{-1}^i & (y=0) \\ \frac{\rho^{t+1}-1}{r}(W_0^i - C_0^i) + \left(\sum_{j=0}^{t}(\theta^{t-j}-1)\rho^j\right)D^i + \rho^{t+1}V_{-1}^i & (y \neq 0) \end{cases}$$

(5.12)

with

$$\rho = 1 + r, \quad \theta = 1 + y$$

(5.13)

and

$$D^i = \alpha^i\left(1 - \frac{r}{y}\right)S_0 + (\alpha^i - \beta^i)C_0$$

(5.14)

for all $i = 1, \dots, N$.

Proof. In the case of $r = 0$ and arbitrary $y > -1$ we have $Y_0 = W_0$ and we conclude from proposition 6, that

$$W_t = (1+y)^t W_0.$$

Thus, we deduce from (5.2), (5.9), and (5.10)

$$S_t^i = \alpha^i W_t - \beta^i C_t$$

$$= (1+y)^t(\alpha^i W_0 - \beta^i C_0)$$

(5.15)

$$= (1+y)^t S_0^i$$

and

$$V_t^i = V_{-1}^i + \sum_{j=0}^{t} S_j^i.$$

This leads to (5.11).

For arbitrary $r > -1$ and $y = 0$ we find with (5.6), (5.9), and (5.10) the recursion

$$V_t^i = (\alpha^i W_t - \beta^i C_t) + (1+r)V_{t-1}^i$$

$$= (W_0^i - C_0^i) - t\alpha^i r S_0 + (1+r)V_{t-1}^i.$$

Inductively follows:

$$V_t^i = (W_0^i - C_0^i)\sum_{j=0}^{t}\rho^j - \alpha^i r S_0 \sum_{j=0}^{t-1}(t-j)\rho^j + \rho^{t+1}V_{-1}^i.$$

(5.16)

For $r \neq 0$ this coincides with the first line of (5.12) and for $r = 0$ this reduces to the first line of (5.11).

Finally we consider the case $r \neq 0$ and $y \neq 0$. With (5.9), (5.10), and (5.13) we find the recursion:

$$V_t^i = (\alpha^i W_t - \beta^i C_t) + (1+\mathrm{r})V_{t-1}^i$$

(5.17)

$$= \alpha^i A + (\alpha^i B - \beta^i C_0)(1+y)^t + (1+r)V_{t-1}^i$$

$$= A^i + \theta^t D^i + \rho V_{t-1}^i$$

with

$$A^i = \alpha^i A = \alpha^i r\left(\frac{S_0}{y} - V_{-1}\right)$$

$$D^i = \alpha^i B - \beta^i C_0 = \alpha^i\left(1 - \frac{r}{y}\right)S_0 + (\alpha^i - \beta^i)C_0.$$

Inductively we find:

$$V_t^i = A^i \sum_{j=0}^{t} \rho^j + D^i \sum_{j=0}^{t} \theta^{t-j}\rho^j + \rho^{t+1}V_{-1}^i \tag{5.18}$$

$$= \frac{\rho^{t+1}-1}{r}(W_0^i - C_0^i) + D^i \sum_{j=0}^{t} (\theta^{t-j}-1)\rho^j + \rho^{t+1}V_{-1}^i.$$

The last line of (5.18) follows from:

$$A^i + D^i = \alpha^i\left(\frac{r}{y}S_0 - rV_{-1}\right) + \alpha^i\left(1 - \frac{r}{y}\right)S_0 + (\alpha^i - \beta^i)C_0 \tag{5.19}$$

$$= \alpha^i(S_0 - R_0) + (\alpha^i - \beta^i)C_0$$

$$= \alpha^i(W_0 - C_0) + (\alpha^i - \beta^i)C_0$$

$$= \alpha^i W_0 - \beta^i C_0.$$

For $r = 0$, proposition 8 leads to simple formulas for the wealth of the household groups of a constant parameter economy. (5.11) parallels (5.5), and if initial wage W_0^i is larger than initial consumption C_0^i of the i-th group, i.e. if $S_0^i \geq 0$, then indebtedness never occurs and each group participates in economic growth.

Proposition 8 showed that in the case of $r = 0$ simple formulas may be derived for the evolution of the wealth of individual household groups. Now we will simplify the formulas in (5.12) for $r \neq 0$ and begin with two lemmas.

Lemma 9. *For* $r \neq 0$ *and* $y \neq r$

$$\sum_{j=0}^{t} (\theta^{t-j}-1)\rho^j = \frac{1}{r}\frac{r\theta^{t+1} - y\rho^{t+1}}{y-r} + \frac{1}{r}. \tag{5.20}$$

Proof. With (5.13) we calculate

$$\sum_{j=0}^{t} (\theta^{t-j}-1)\rho^j = \theta^t \sum_{j=0}^{t} \left(\frac{\rho}{\theta}\right)^j - \sum_{j=0}^{t} \rho^j$$

$$= \theta^t \frac{\left(\frac{\rho}{\theta}\right)^{t+1} - 1}{\frac{\rho}{\theta} - 1} - \frac{\rho^{t+1}-1}{\rho-1}$$

$$= \frac{\rho^{t+1} - \theta^{t+1}}{\rho - \theta} - \frac{\rho^{t+1}-1}{\rho-1}$$

$$= \frac{r\rho^{t+1} - r\theta^{t+1} - (r-y)\rho^{t+1}}{r(r-y)} + \frac{1}{r},$$

which was to be shown.

Lemma 10. *For* $r \neq 0$

$$\sum_{j=0}^{t} j\rho^j = \frac{t\rho^{t+2} - (t+1)\rho^{t+1} + \rho}{(\rho-1)^2}. \tag{5.21}$$

Proof. For $t = 0$ the left and right hand side both equal zero. Assume (5.21) is already proved for some t. Then

$$\sum_{j=0}^{t+1} j\rho^j = \sum_{j=0}^{t} j\rho^j + (t+1)\rho^{t+1}$$

$$= \frac{t\rho^{t+2} - (t+1)\rho^{t+1} + \rho}{(\rho-1)^2} + \frac{(\rho-1)^2(t+1)\rho^{t+1}}{(\rho-1)^2}$$

$$= \frac{(t+1)\rho^{t+3} + (t-2t-2)\rho^{t+2} + \rho}{(\rho-1)^2},$$

which was to be shown. Alternatively, (5.21) may also be proved with $s_n = \sum_{j=0}^{n} q^j = \frac{q^{n+1}-1}{q-1}$ and calculation of $q\frac{d}{dq}s_n$.

Proposition 11. *In a constant parameter economy we have for $r \neq 0$ and $i = 1, \dots, N$*

$$V_t^i = \begin{cases} \rho^{t+1}\frac{1}{r}\Delta^i + (t+1)\alpha^i S_0 - \frac{1}{r}\Delta^i + V_{-1}^i & (y=0) \\ \rho^{t+1}\left(\frac{1}{r-y}\Delta_1^i + \Delta_2^i\right) + \theta^{t+1}\left(\frac{1}{y-r}\Delta_1^i + \frac{\alpha^i}{y}S_0\right) + \xi^i & (y \neq 0, y \neq r) \\ \rho^{t+1}\left(\left((t+1)\frac{1}{\rho} - \frac{1}{r}\right)\Delta_1^i + \frac{1}{r}S_0^i\right) + \xi^i & (y = r) \end{cases} \tag{5.22}$$

with

$$\xi^i = \alpha^i\left(V_{-1} - \frac{S_0}{y}\right)$$

and

$$\Delta_1^i = (\alpha^i - \beta^i)C_0$$

$$\Delta_2^i = V_{-1}^i - \alpha^i V_{-1}$$

$$\Delta^i = S_0^i - \alpha^i S_0 \tag{5.23}$$

$$= (\alpha^i - \beta^i)C_0 + r(V_{-1}^i - \alpha^i V_{-1})$$

$$= \Delta_1^i + r\Delta_2^i.$$

Proof. First, we assume $y = 0$. With Lemma 10 we find:

$$\sum_{j=0}^{t-1}(t-j)\rho^j = t\frac{\rho^t - 1}{r} - \frac{(t-1)\rho^{t+1} - t\rho^t + \rho}{r^2}$$

$$= \frac{\rho^{t+1} - \rho - tr}{r^2}.$$

Insertion into the first line of (5.12) leads to:

$$V_t^i = \frac{\rho^{t+1}-1}{r}(W_0^i - C_0^i) - \frac{\rho^{t+1} - \rho - tr}{r}\alpha^i S_0 + \rho^{t+1}V_{-1}^i \tag{5.24}$$

$$= \frac{\rho^{t+1}}{r}(W_0^i - C_0^i - \alpha^i S_0 + rV_{-1}^i) + t\alpha^i S_0$$

$$-\frac{1}{r}(W_0^i - C_0^i - \rho\alpha^i S_0).$$

With $S_0^i = W_0^i + R_0^i - C_0^i$ and $R_0^i = rV_{-1}^i$ we write:

$$W_0^i - C_0^i - \alpha^i S_0 + rV_{-1}^i = S_0^i - \alpha^i S_0 = \Delta^i.$$

The second equality of (5.23) follows from:

$$W_0^i - C_0^i - \alpha^i S_0 + rV_{-1}^i = \alpha^i W_0 - \beta^i C_0 - \alpha^i(W_0 + rV_{-1} - C_0) + rV_{-1}^i$$

$$= (\alpha^i - \beta^i)C_0 + r(V_{-1}^i - \alpha^i V_{-1}).$$

Furthermore,

$$W_0^i - C_0^i - \rho\alpha^i S_0 = S_0^i - R_0^i - (1+r)\alpha^i S_0$$

$$= (S_0^i - \alpha^i S_0) - r(V_{-1}^i + \alpha^i S_0)$$

$$= \Delta^i - r(V_{-1}^i + \alpha^i S_0),$$

and we get the first line of (5.22).

Now we consider the case $y \neq 0$ and $y \neq r$. For the second line of (5.12) we use lemma 9 and (5.14) to find

$$V_t^i = \frac{\rho^{t+1}-1}{r}(W_0^i - C_0^i) + \left(\frac{1}{r}\frac{r\theta^t - y\rho^{t+1}}{y-r} + \frac{1}{r}\right)D^i + \rho^{t+1}V_{-1}^i$$

$$= \frac{\rho^{t+1}}{r}\left(W_0^i - C_0^i - \frac{yD^i}{y-r} + rV_{-1}^i\right) + \theta^{t+1}\frac{D^i}{y-r} + \frac{D^i - W_0^i + C_0^i}{r}.$$

We have $D^i = \alpha^i \frac{y-r}{y} S_0 + \Delta_1^i$, so that:

$$W_0^i - C_0^i - \frac{y}{y-r}D^i + rV_{-1}^i = S_0^i - \alpha^i S_0 - \frac{y}{y-r}\Delta_1^i$$

$$= \Delta_1^i + r\Delta_2^i - \frac{y}{y-r}\Delta_1^i$$

$$= -\frac{r}{y-r}\Delta_1^i + r\Delta_2^i$$

and

$$\frac{D^i}{y-r} = \frac{1}{y-r}\Delta_1^i + \frac{\alpha^i}{y}S_0.$$

With (5.19) we write $D^i - W_0^i + C_0^i = -A^i = \alpha^i r\left(V_{-1} - \frac{S_0}{y}\right)$, so that:

$$V_t^i = \rho^{t+1}\left(-\frac{1}{y-r}\Delta_1^i + \Delta_2^i\right) + \frac{1}{y}\theta^{t+1}\left(\frac{y}{y-r}\Delta_1^i + \alpha^i S_0\right) + \alpha^i\left(V_{-1} - \frac{S_0}{y}\right),$$

and this is the second line of (5.22).

Finally we consider $r = y \neq 0$. Now (5.14) reduces to

$$D^i = (\alpha^i - \beta^i)C_0 = \Delta_1^i.$$

Further, with $\rho = \theta$ we have $\sum_{j=0}^{t}\left(\theta^{t-j} - 1\right)\rho^j = (t+1)\rho^t - \frac{\rho^{t+1}-1}{r}$, and the second line of (5.12) may be written as:

$$V_t^i = \frac{\rho^{t+1}-1}{r}\alpha^i(W_0 - C_0) + (t+1)\rho^t\Delta_1^i + \rho^{t+1}V_{-1}^i \tag{5.25}$$

$$= \frac{\rho^{t+1}}{r}\left(\alpha^i(W_0 - C_0) + (t+1)\frac{r}{\rho}\Delta_1^i + R_0^i\right) - \frac{\alpha^i}{r}(W_0 - C_0).$$

but

$$\alpha^i(W_0 - C_0) + R_0^i = W_0^i + R_0^i - C_0^i - (\alpha^i - \beta^i)C_0$$

$$= S_0^i - \Delta_1^i.$$

If we insert this into (5.25), we establish:

$$V_t^i = \frac{\rho^{t+1}}{r}\left(S_0^i + \Delta_1^i\left((t+1)\frac{r}{\rho} - 1\right)\right) - \frac{\alpha^i}{r}(W_0 - C_0).$$

With $Y_0 = C_0 + S_0 = W_0 + R_0 = W_0 + rV_{-1}$ and:

$$\frac{\alpha^i}{r}(C_0 - W_0) = \frac{\alpha^i}{r}(R_0 - S_0) = \alpha^i\left(V_{-1} - \frac{S_0}{r}\right)$$

we obtain the third line of (5.22), and the proposition is proved.

If $V_{-1} > 0$, the second line in (5.23) may be written as

$$\Delta_2^i = (v^i - \alpha^i)V_{-1}, \quad v^i = \frac{V_{-1}^i}{V_{-1}}.$$

Polarization

How does an economy evolve in the long run? Are there parameter constellations such that the wealth of the household groups does not behave too differently? What are the prerequisites for the wealth of the households becoming more and more different? Here we analyze these questions for the constant parameter economy.

Definition 12: *We say that polarization eventually occurs in the economy, if there are household groups* $1 \leq k, l \leq N$ *so that:*

$$\lim_{t\to\infty} V_t^k = +\infty \qquad and \qquad V_t^l \leq c$$

for some constant c or if:

$$\lim_{t\to\infty} V_t^i = +\infty \ \ for\ all\ \ 1 \leq \ \ i \leq \ \ N,$$

$$but \ \lim_{t\to\infty} \frac{V_t^k}{V_t^l} = +\infty$$

If $y \geq 0$ and $S_0 > 0$, then proposition 1 shows that $\lim_{t\to\infty} V_t = \infty$, so there is at least one household group k with $\lim_{t\to\infty} V_t^k = \infty$.

By proposition 8 and proposition 11 the wealth of a household group $i = 1, \dots, N$ is given by

$$V_t^i = \begin{cases} (t+1)S_0^i + V_{-1}^i & (r=0, y=0) \\ \theta^{t+1}\frac{1}{y}S_0^i + \left(V_{-1}^i - \frac{1}{y}S_0^i\right) & (r=0, y\neq 0) \\ \rho^{t+1}\frac{1}{r}\Delta^i + (t+1)\alpha^i S_0 + \left(V_{-1}^i - \frac{1}{r}\Delta^i\right) & (r\neq 0, y=0) \\ \rho^{t+1}\left(\frac{1}{r-y}\Delta_1^i + \Delta_2^i\right) + \theta^{t+1}\left(\frac{1}{y-r}\Delta_1^i + \frac{\alpha^i}{y}S_0\right) + \xi^i & (r, y\neq 0, y\neq r) \\ \rho^{t+1}\left(\left((t+1)\frac{1}{\rho} - \frac{1}{r}\right)\Delta_1^i + \frac{1}{r}S_0^i\right) + \xi^i & (r\neq 0, y=r). \end{cases} \tag{5.26}$$

Lemma 13. *For each household group* $1 \leq i \leq N$ *we either have* $V_t^i = c$ *for some constant* c *or* $lim_{t\to\infty} V_t^i = \pm\infty$.

Proof. This follows from (5.26) and

$$a^t \to \begin{cases} 0 & (0 < a < 1) \\ 1 & (a = 1) \\ \infty & (a > 1) \end{cases}$$

for $t \to \infty$.

We observe that if:

$$\lambda\Delta_1^k + \mu\Delta_2^k > 0$$

for some $1 \leq k \leq N$, where $\lambda, \mu \in \mathbb{R}$, then there is some $1 \leq l \leq N$ with

$$\lambda\Delta_1^l + \mu\Delta_2^l < 0,$$

because $\sum_{i=1}^N \Delta_1^i = \sum_{i=1}^N \Delta_2^i = 0$.

To restrict the number of different cases, we assume $r \geq 0$ and $y \geq 0$ for the remainder of this section.

Proposition 14. *Assume a constant parameter economy with* $V_{-1} > 0$ *and* $S_0 > 0$. *Further assume that there are two household groups* k, l *with*

$$\begin{gathered} v^k > \alpha^k > \beta^k \\ v^l < \alpha^l < \beta^l. \end{gathered} \tag{5.27}$$

1. If $r > 0$ and $y \leq r$, then polarization occurs.
2. If $y > r > 0$ and if

$$\frac{1}{y-r}\Delta_1^l + \frac{\alpha^l}{y} S_0 \leq 0, \tag{5.28}$$

then the economy becomes polarized.

Proof. (5.27) implies

$$\Delta_1^k > 0, \quad \Delta_2^k > 0,$$

$$\Delta_1^l < 0, \quad \Delta_2^l < 0.$$

1. Case $r > 0$ and $y \leq r$: This implies $\rho \geq \theta$, and from the last three lines of (5.26) we derive
$$\lim_{t\to\infty} V_t^k = +\infty, \quad \lim_{t\to\infty} V_t^l = -\infty,$$
so that polarization occurs.
2. Case $0 < r < y$: Now $\theta > \rho$ and from the fourth line of (5.26) we conclude
$$V_t^i = \theta^{t+1}\left(\frac{1}{y-r}\Delta_1^i + \frac{\alpha^i}{y}S_0\right) + \rho^{t+1}\left(\frac{1}{r-y}\Delta_1^i + \Delta_2^i\right) + \xi^i.$$

By assumption, $\Delta_1^k > 0$, so that

$$\lim_{t\to\infty} V_t^k = +\infty.$$

Assume that (5.28) holds for some household group l.

a) If $\frac{1}{y-r}\Delta_1^l + \frac{\alpha^l}{y}S_0 < 0$, then we find
$$\lim_{t\to\infty} V_t^l = -\infty$$

and polarization occurs.

b) Finally, we assume
$$\frac{1}{y-r}\Delta_1^l + \frac{\alpha^l}{y}S_0 = 0$$

i. If additionally $\frac{1}{r-y}\Delta_1^l + \Delta_2^l < 0$, then we are again led to
$$\lim_{t\to\infty} V_t^l = -\infty$$
and polarization occurs.

ii. If
$$\frac{1}{r-y}\Delta_1^l + \Delta_2^l = 0$$
then $V_t^l = \xi^l$ and the economy becomes polarized.

iii. If
$$\frac{1}{r-y}\Delta_1^i + \Delta_2^i > 0$$
then we have
$$\lim_{t\to\infty} V_t^l = \infty$$
but
$$\lim_{t\to\infty} \frac{V_t^k}{V_t^l} = \infty,$$
and again polarization occurs.

Definition 15: *The conditions*

$$\upsilon^k > \alpha^k > \beta^k \tag{5.29}$$

characterize a wealthy household group. Its share of wealth is larger than its share of labor income. And its share of labor income is greater than its share of consumption. A household group with properties (5.29) will be called an *upper household group*.

On the contrary, the conditions

$$\upsilon^l < \alpha^l < \beta^l \tag{5.30}$$

define households with shares in wealth and labor income being less than their shares in consumption. Here we note that consumption cannot be reduced arbitrarily. A household group with properties (5.30) will be called a *lower household group*.

Proposition 16. *In a constant parameter economy polarization will be avoided if:*

1. $r = 0$: $S_0^i > 0$ *for all* i *or* $S_0^i = 0$ *for all* i
2. $r > y = 0$: $\Delta_1^i + r\Delta_2^i = 0$ *and* $\alpha^i > 0$ *for all* i
3. $r > y > 0$: $\frac{1}{r-y}\Delta_1^i + \Delta_2^i = 0$ *and* $\frac{1}{y-r}\Delta_1^i + \frac{\alpha^i}{y}S_0 > 0$ *for all* i
4. $r = y > 0$: $\Delta_1^i = 0$ *for all* i
5. $y > r > 0$: $\frac{1}{y-r}\Delta_1^i + \frac{\alpha^i}{y}S_0 > 0$ *for all* i.

Proof. These assertions follow from (5.26).

Conditions 2., 3., and 4. in proposition 16 require special parameter constellations for the whole economy that do not seem suitable to describe realistic situations. Conditions 1. and 5. in contrast have the following interpretations:

In the case of $r = 0$ no polarization occurs if each household or no household saves.

In the case of $y > r > 0$ no polarization occurs, if for each group $1 \leq i \leq N$

$$\frac{1}{y-r}\Delta_1^i + \frac{\alpha^i}{y}S_0 > 0, \tag{5.31}$$

and in this case $\lim_{t\to\infty} V_t^i = \infty$ for all i. Because of

$$\sum_{i=1}^{N}\left(\frac{1}{y-r}\Delta_1^i + \frac{\alpha^i}{y}S_0\right) = \frac{1}{y}S_0 > 0, \tag{5.32}$$

relation (5.31) is fulfilled for at least one household group. But Proposition 14 shows that if (5.31) is violated for at least one household group l, then $\lim_{t\to\infty} V_t^l = -\infty$, and polarization occurs.

To avoid polarization in a constant parameter economy, there are only two possibilities:

- Either capital income is avoided, $r = 0$,

- or permanent growth with growth rates above the capital income rate, $y > r > 0$, must be established and (5.31) has to be fulfilled for each household group.

To put it differently: if capital returns are admitted unrestrictedly, and this implies $r > 0$ for a constant parameter economy, then permanent growth $y > r > 0$ above the capital return rate has to be generated to avoid polarization; but even in this case, the additional requirement (5.31) has to be fulfilled for each group.

Interest Transfer

Interest transfer Z_t^i is given by

$$\begin{aligned} Z_t^i &= R_t^i - \frac{C_t^i}{C_t}R_t \\ &= R_t^i - \beta^i R_t \\ &= r\left(V_{t-1}^i - \beta^i V_{t-1}\right). \end{aligned} \tag{5.33}$$

For $r = 0$, there is obviously no interest transfer. For $r > 0$ we assume the prerequisites of proposition 14. From (5.26), (5.33), and proposition 1 we conclude

$$Z_t^i = \begin{cases} \rho^t \Delta^i + tr(\alpha^i - \beta^i)S_0 - \Delta_1^i + (\alpha^i - \beta^i)rV_{-1} & (r > y = 0) \\ \rho^t r\left(\frac{1}{r-y}\Delta_1^i + \Delta_2^i\right) + \theta^t r\left(\frac{1}{y-r}\Delta_1^i + \frac{\alpha^i - \beta^i}{y}S_0\right) + \gamma^i & (r > y > 0) \\ \rho^t\left(\left(t\frac{r}{\rho} - 1\right)\Delta_1^i + S_0^i - \beta^i S_0\right) + \gamma^i & (r = y > 0) \\ \theta^t r\left(\frac{1}{y-r}\Delta_1^i + \frac{\alpha^i - \beta^i}{y}S_0\right) + \rho^t r\left(\frac{1}{r-y}\Delta_1^i + \Delta_2^i\right) + \gamma^i & (y > r > 0), \end{cases} \tag{5.34}$$

where γ^i is defined by

$$\gamma^i = r\xi^i - r\beta^i\left(V_{-1} - \frac{S_0}{y}\right) = (\alpha^i - \beta^i)r\left(V_{-1} - \frac{S_0}{y}\right).$$

If k is an upper household group, then $\Delta_1^k > 0$, $\Delta_2^k > 0$, and $\Delta^k = \Delta_1^k + r\Delta_2^k > 0$. If l is a lower household group, then $\Delta_1^l < 0$, $\Delta_2^l < 0$, and $\Delta^l = \Delta_1^l + r\Delta_2^l < 0$. Therefore we conclude

$$Z_t^k \to \infty, \quad Z_t^l \to -\infty \quad (t \to \infty)$$

for each case in (5.34). Irrespective of the relation between $r > 0$ and $y \geq 0$ we observe unrestricted interest transfer payments. Because of $\sum_{i=1}^N Z_t^i = 0$ interest transfer characterizes a redistribution mechanism. The increase of wealth of upper household groups due to capital income is financed by other household groups.

Relative Change of Wealth of the Household Group

Corollary 17. *Assume $y \geq 0$ and $r \geq 0$. For the relative change of wealth of the i-th household group we have asymptotically*

$$\lim_{t\to\infty} \frac{V_t^i - V_{t-1}^i}{V_{t-1}^i} = \max(r, y),$$

where we additionally assume

$$\begin{aligned} & S_0^i \neq 0 && (r = 0) \\ & \Delta^i \neq 0 && (r > y = 0) \\ & \frac{1}{r-y}\Delta_1^i + \Delta_2^i \neq 0 && (r > y > 0) \\ & \Delta_1^i \neq 0 && (r = y > 0) \\ & \frac{1}{y-r}\Delta_1^i + \frac{\alpha^i}{y}S_0 \neq 0 && (y > r > 0). \end{aligned}$$

Proof. We step through the different cases in (5.26) and use $\theta^{t+1} - \theta^t = \theta^t(\theta - 1) = \theta^t y$ as well as $\rho^{t+1} - \rho^t = \rho^t r$.

1. $r = 0$ and $y = 0$. The first case in (5.26) leads to

$$\frac{V_t^i - V_{t-1}^i}{V_{t-1}^i} = \frac{S_0^i}{V_{-1}^i + tS_0^i} \to 0 \quad (t \to \infty)$$

2. $r = 0$ and $y > 0$. The second case in (5.26) leads to

$$\frac{V_t^i - V_{t-1}^i}{V_{t-1}^i} = \frac{yS_0^i}{\frac{yV_{-1}^i - S_0^i}{\theta^t} + S_0^i} \to y \quad (t \to \infty).$$

3. $r > 0$ and $y = 0$. The third case in (5.26) leads to

$$\frac{V_t^i - V_{t-1}^i}{V_{t-1}^i} = r\frac{\Delta^i + \frac{1}{\rho^t}\alpha^i S_0}{\Delta^i + \frac{r}{\rho^t}(t\alpha^i S_0 + \xi^i)}$$

$$\to r \quad (t \to \infty).$$

4. $r > 0$ and $y > 0$ and $y \neq r$. In this case we have

$$\frac{V_t^i - V_{t-1}^i}{V_{t-1}^i}$$
$$= \frac{\rho^t r\left(\frac{1}{r-y}\Delta_1^i + \Delta_2^i\right) + \theta^t y\left(\frac{1}{y-r}\Delta_1^i + \frac{\alpha^i}{y}S_0\right)}{\rho^t\left(\frac{1}{r-y}\Delta_1^i + \Delta_2^i\right) + \theta^t\left(\frac{1}{y-r}\Delta_1^i + \frac{\alpha^i}{y}S_0\right) + \xi^i}.$$

We conclude

$$\lim_{t\to\infty}\frac{V_t^i - V_{t-1}^i}{V_{t-1}^i} = \begin{cases} r & if \quad r > y \\ y & if \quad r < y. \end{cases}$$

5. $y = r > 0$. In this case we have

$$\frac{V_t^i - V_{t-1}^i}{V_{t-1}^i} = \frac{r\left(\left(t\frac{1}{\rho} - \frac{1}{r}\right)\Delta_1^i + \frac{1}{r}S_0^i\right) + \Delta_1^i}{\left(\left(t\frac{1}{\rho} - \frac{1}{r}\right)\Delta_1^i + \frac{1}{r}S_0^i\right) + \frac{1}{\rho^t}\xi^i}$$

$$\to r \quad (t \to \infty).$$

Thus, relative change of wealth of individual household groups is dependent on the capital return rate r and on economic growth y. If $r \geq y$, then Corollary 17 above shows that $\frac{V_t^i - V_{t-1}^i}{V_{t-1}^i} \approx r$ for large t, but by Corollary 5 we know that the relative change of total wealth is approximately $\frac{V_t - V_{t-1}}{V_{t-1}} \approx y$ for t large enough. There is no contradiction here, because if $\frac{V_t^i - V_{t-1}^i}{V_{t-1}^i}$ is approximately r for every i, then this is in particular true for those groups that eventually become indebted, i.e., whose wealth becomes negative. If a household group has negative and strictly decreasing evolution of wealth, then the definition of relative change leads to positive values. Thus, while the relative change of total wealth is y asymptotically, the relative change of the positive wealth V_t^i of the upper groups is $r > y$, and this is compensated, i.e. financed, by the indebtedness of other household groups.

Numerical Simulations

Dynamic analysis may be formulated as a dynamical system, and this allows for efficient numerical simulations in the constant parameter case.

Proposition 18. *Let V_t^i be the wealth of the ith household group at time t. Then*

$$V_t^i = \sum_{j=1}^{N} a_t^{ij} V_{t-1}^j + b_t^i, \tag{6.1}$$

where

$$a_t^{ij} = (1 + r_t)\delta^{ij} - r_t\alpha_t^i \tag{6.2}$$

$$b_t^i = \alpha_t^i S_t + \left(\alpha_t^i - \beta_t^i\right)C_t.$$

Here, $\delta^{ij} = 1$ for $i = j$ and $\delta^{ij} = 0$ for $i \neq j$ denotes the Kronecker symbol.

Proof. The assertion follows from

$$V_t^i = V_{t-1}^i + S_t^i$$

$$= V_{t-1}^i + W_t^i + R_t^i - C_t^i$$

$$= V_{t-1}^i + \alpha_t^i (Y_t - r_t V_{t-1}) + r_t V_{t-1}^i - C_t^i$$

$$= (1 + r_t) V_{t-1}^i - r_t \alpha_t^i \sum_{j=1}^{N} V_{t-1}^j + \left(\alpha_t^i Y_t - C_t^i\right).$$

We check

$$\sum_{i=1}^{N} a_t^{ij} = 1 + r_t - r_t \sum_{i=1}^{N} \alpha_t^i = 1$$

and

$$\sum_{i=1}^{N} b_t^i = S_t$$

,

so that

$$V_t = \sum_{i=1}^{N} V_t^i = \sum_{j=1}^{N} \left(\sum_{i=1}^{N} a_t^{ij} \right) V_{t-1}^j + \sum_{i=1}^{N} b_t^i$$

$$= V_{t-1} + S_t,$$

as expected.

Corollary 19. *We have*

$$\mathbf{V}_t = \mathbf{A}_t \mathbf{V}_{t-1} + \mathbf{b}_t, \tag{6.3}$$

where

$$\mathbf{V}_t = \begin{pmatrix} V_t^1 \\ \vdots \\ V_t^N \end{pmatrix},$$

$$\mathbf{A}_t = \begin{pmatrix} a_t^{11} & \cdots & a_t^{1N} \\ \vdots & & \vdots \\ a_t^{N1} & \cdots & a_t^{NN} \end{pmatrix}$$

and

$$\mathbf{b}_t = \begin{pmatrix} b_t^1 \\ \vdots \\ b_t^N \end{pmatrix}.$$

Proof. With (6.2), this follows immediately from recursion (6.1).

(6.3) is the recursion equation for a time discrete inhomogeneous dynamical system with initial value $\mathbf{V}_{-1}$.

The Constant Parameter Economy as a Dynamic System

In a constant parameter case, (6.2) restricts to

$$a^{ij} = a_t^{ij} = \delta_{ij} + r\left(\delta_{ij} - \alpha^i\right),$$

that is

$$\mathbf{A} = \mathbf{A}_t =$$

$$\begin{pmatrix} 1 + r(1 - \alpha^1) & -r\alpha^1 & \cdots & -r\alpha^1 \\ -r\alpha^2 & 1 + r(1 - \alpha^2) & \cdots & -r\alpha^2 \\ \vdots & & \ddots & \vdots \\ -r\alpha^N & \cdots & & 1 + r(1 - \alpha^N) \end{pmatrix}$$

$$= I + r\left(I - \begin{pmatrix} \alpha^1 & \cdots & \alpha^1 \\ \vdots & & \vdots \\ \alpha^N & \cdots & \alpha^N \end{pmatrix} \right),$$

where I denotes the $N \times N$-identity matrix, and

$$b_t^i = \left(\alpha^i S_0 + \left(\alpha^i - \beta^i\right) C_0\right)(1 + y)^t.$$

Thus, for the case of a constant parameter economy we get the dynamical system

$$\mathbf{V}_t = \mathbf{A}\mathbf{V}_{t-1} + \mathbf{b}_t$$

with

$$\mathbf{b}_t = \mathbf{b}_0(1+y)^t$$

and initial value

$$\mathbf{V}_{-1} = \begin{pmatrix} V_{-1}^1 \\ \vdots \\ V_{-1}^N \end{pmatrix}.$$

A Scilab program for the dynamics if a constant parameter economy

In this section we present Scilab code for the implementation of the dynamical system of section 6.1.

Numerical Results

In this section we show the results of the code of the previous section for the parameter constellation

$$r = 2\%, \quad y = 5\%$$

and

$$r = 2\%, \quad y = 1\%.$$

<u>Results for r = 2% and y = 5%</u>

In this first case the wealth of each household group increases over time, and each member of the economy might feel that “things are looking up”. However, massive redistribution of wealth occurs, but nevertheless no polarization is visible, because economic growth is large enough to disguise interest transfer effects.

10.	17.	46.	61.	83.	114.	172.	245.	451.	1220.
10.	17.	47.	62.	87.	120.	180.	257.	473.	1275.
10.	18.	48.	63.	91.	127.	189.	269.	496.	1334.
11.	18.	49.	65.	95.	134.	199.	283.	520.	1394.
11.	18.	50.	66.	99.	141.	209.	296.	546.	1458.
11.	19.	51.	68.	103.	149.	219.	311.	573.	1525.
11.	19.	52.	69.	108.	157.	230.	326.	601.	1596.
12.	20.	53.	70.	113.	166.	241.	342.	631.	1670.
12.	20.	54.	72.	118.	175.	253.	359.	662.	1747.
12.	21.	56.	74.	124.	185.	266.	377.	695.	1828.
13.	21.	57.	75.	129.	195.	279.	396.	729.	1913.
13.	22.	58.	77.	136.	205.	293.	415.	765.	2003.
13.	22.	59.	79.	142.	217.	308.	436.	803.	2096.
14.	23.	61.	81.	149.	228.	324.	458.	843.	2195.
14.	24.	62.	82.	156.	241.	340.	481.	885.	2298.
14.	24.	64.	84.	163.	254.	357.	505.	929.	2406.
15.	25.	65.	86.	171.	268.	375.	530.	976.	2520.
15.	26.	67.	88.	179.	282.	394.	556.	1024.	2639.
16.	27.	69.	91.	188.	298.	414.	584.	1075.	2764.
16.	28.	70.	93.	197.	314.	435.	613.	1129.	2895.
17.	28.	72.	95.	206.	331.	457.	644.	1185.	3033.
17.	29.	74.	98.	216.	349.	480.	677.	1245.	3177.
18.	30.	76.	100.	227.	368.	504.	711.	1307.	3329.
18.	31.	78.	103.	238.	387.	530.	746.	1372.	3488.
19.	32.	80.	106.	250.	408.	557.	784.	1441.	3654.
20.	34.	82.	108.	262.	430.	585.	823.	1513.	3830.
20.	35.	85.	111.	275.	453.	615.	865.	1589.	4013.
21.	36.	87.	114.	288.	478.	646.	908.	1669.	4206.
22.	37.	90.	117.	302.	503.	679.	954.	1752.	4409.
23.	39.	92.	121.	317.	530.	714.	1002.	1840.	4621.
23.	40.	95.	124.	333.	558.	750.	1053.	1933.	4844.
24.	42.	98.	128.	349.	588.	788.	1106.	2029.	5078.
25.	43.	100.	131.	367.	620.	828.	1161.	2131.	5323.
26.	45.	104.	135.	385.	653.	870.	1220.	2238.	5581.
27.	47.	107.	139.	404.	687.	914.	1282.	2351.	5851.
28.	49.	110.	143.	424.	724.	961.	1346.	2469.	6135.
29.	51.	113.	147.	445.	762.	1010.	1414.	2593.	6433.
30.	53.	117.	152.	467.	802.	1061.	1486.	2723.	6746.
32.	55.	121.	157.	491.	845.	1115.	1561.	2860.	7074.
33.	57.	124.	161.	515.	889.	1172.	1639.	3003.	7418.
34.	59.	128.	166.	541.	936.	1231.	1722.	3154.	7780.
36.	62.	133.	172.	568.	986.	1294.	1809.	3313.	8159.
37.	65.	137.	177.	596.	1037.	1360.	1900.	3479.	8557.
39.	67.	141.	183.	626.	1092.	1429.	1996.	3654.	8975.
40.	70.	146.	189.	657.	1149.	1502.	2097.	3838.	9414.
42.	73.	151.	195.	690.	1209.	1578.	2203.	4031.	9874.
44.	77.	156.	201.	725.	1273.	1658.	2314.	4233.	10358.
46.	80.	162.	208.	761.	1339.	1742.	2431.	4446.	10865.
48.	83.	167.	215.	799.	1409.	1831.	2554.	4669.	11398.
50.	87.	173.	222.	839.	1483.	1924.	2683.	4904.	11957.
52.	91.	179.	230.	882.	1560.	2021.	2818.	5150.	12544.
54.	95.	186.	238.	926.	1641.	2124.	2960.	5409.	13160.

Figure 3: Wealth of each of the 10 household groups of the economy for $r = 2\%$ and $y = 5\%$.

Each column contains the wealth of a houshold, each row contains the wealth of all groups for a year from -1 to 50.

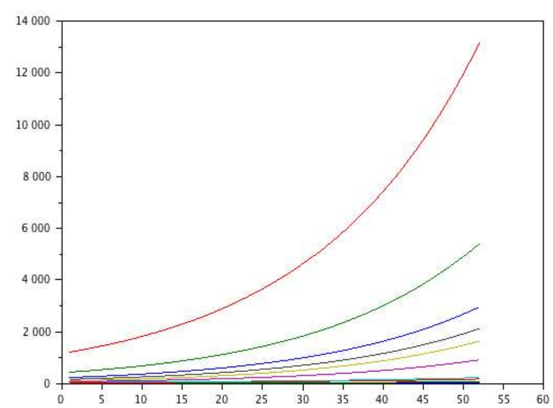

Figure 4: Graphical output of the Scilab code

above for the evolution of wealth in a constant parameter economy with $r = 2\%$ and $y = 5\%$.

<u>*Results for r = 2% and y = 1%*</u>

Here, economic growth is low, in particular lower than the mean capital return rate. Now, polarization occurs and the economy becomes more and more unstable.

10.	17.	46.	61.	83.	114.	172.	245.	451.	1220.
10.	17.	47.	62.	87.	120.	180.	257.	473.	1275.
10.	18.	48.	63.	90.	127.	189.	269.	495.	1332.
10.	18.	48.	64.	94.	133.	197.	281.	518.	1389.
11.	18.	49.	65.	97.	139.	206.	293.	541.	1448.
11.	18.	50.	66.	101.	145.	215.	305.	564.	1508.
10.	18.	50.	67.	104.	152.	223.	318.	588.	1568.
10.	18.	50.	67.	107.	158.	232.	330.	612.	1630.
10.	17.	51.	68.	111.	164.	240.	343.	636.	1694.
10.	17.	51.	68.	114.	171.	249.	355.	661.	1758.
10.	16.	51.	68.	117.	177.	258.	368.	686.	1823.
9.	16.	51.	68.	120.	183.	266.	381.	711.	1890.
9.	15.	50.	68.	123.	189.	275.	394.	737.	1958.
9.	14.	50.	68.	126.	195.	284.	407.	763.	2028.
8.	13.	50.	67.	129.	202.	293.	420.	789.	2098.
8.	12.	49.	67.	132.	208.	301.	433.	816.	2170.
7.	11.	48.	66.	134.	214.	310.	446.	843.	2244.
6.	10.	47.	65.	137.	220.	319.	460.	871.	2318.
5.	8.	46.	64.	139.	226.	328.	473.	899.	2394.
5.	7.	45.	63.	142.	232.	337.	487.	927.	2472.
4.	5.	44.	61.	144.	239.	346.	501.	956.	2551.
3.	3.	42.	60.	146.	245.	355.	514.	985.	2632.
1.	1.	40.	58.	148.	251.	363.	528.	1015.	2714.
0.	- 1.	39.	56.	150.	257.	372.	542.	1045.	2798.
- 1.	- 3.	37.	54.	152.	263.	381.	556.	1075.	2883.
- 2.	- 6.	34.	51.	154.	269.	390.	570.	1106.	2970.
- 4.	- 8.	32.	49.	155.	274.	399.	585.	1138.	3058.
- 5.	- 11.	29.	46.	157.	280.	408.	599.	1170.	3148.
- 7.	- 14.	27.	43.	158.	286.	417.	613.	1202.	3240.
- 9.	- 17.	24.	40.	159.	292.	426.	628.	1235.	3334.
- 10.	- 21.	20.	36.	160.	298.	435.	643.	1268.	3430.
- 12.	- 24.	17.	32.	161.	303.	444.	657.	1302.	3527.
- 14.	- 28.	14.	28.	162.	309.	453.	672.	1336.	3626.
- 17.	- 32.	10.	24.	163.	315.	461.	687.	1371.	3727.
- 19.	- 36.	6.	20.	163.	320.	470.	702.	1406.	3830.
- 21.	- 40.	1.	15.	163.	325.	479.	717.	1442.	3935.
- 24.	- 45.	- 3.	10.	163.	331.	488.	732.	1478.	4042.
- 26.	- 49.	- 8.	5.	163.	336.	497.	748.	1515.	4151.
- 29.	- 54.	- 13.	- 1.	163.	341.	506.	763.	1553.	4262.
- 32.	- 59.	- 18.	- 7.	163.	347.	514.	779.	1591.	4376.
- 35.	- 65.	- 24.	- 13.	162.	352.	523.	794.	1629.	4491.
- 38.	- 70.	- 29.	- 20.	161.	357.	532.	810.	1668.	4609.
- 41.	- 76.	- 35.	- 27.	160.	362.	541.	826.	1708.	4728.
- 45.	- 83.	- 42.	- 34.	159.	366.	549.	841.	1748.	4851.
- 48.	- 89.	- 48.	- 41.	158.	371.	558.	857.	1789.	4975.
- 52.	- 96.	- 55.	- 49.	156.	376.	567.	873.	1831.	5102.
- 56.	- 103.	- 63.	- 57.	154.	381.	575.	890.	1873.	5231.
- 60.	- 110.	- 70.	- 66.	152.	385.	584.	906.	1916.	5363.
- 64.	- 117.	- 78.	- 75.	150.	389.	592.	922.	1959.	5498.
- 68.	- 125.	- 86.	- 84.	147.	394.	601.	939.	2004.	5635.
- 72.	- 133.	- 95.	- 94.	145.	398.	609.	955.	2048.	5774.
- 77.	- 141.	- 104.	- 104.	142.	402.	617.	972.	2094.	5916.

Figure 5: Wealth of each of the 10 household groups of the economy for $r = 2\%$ and $y = 1\%$.

Each column contains the wealth of a houshold, each row contains the wealth of all groups for a year from -1 to 50.

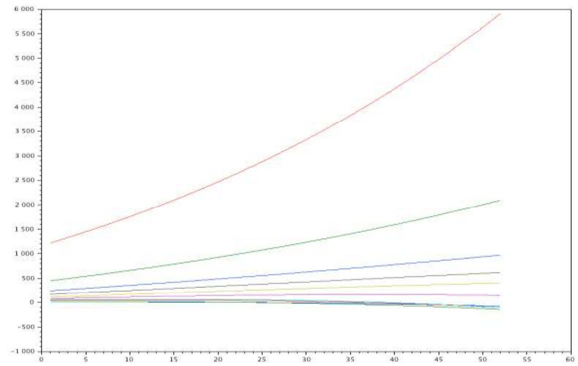

Figure 6: Graphical output of the Scilab code above for the evolution of wealth in a constant parameter economy with $r = 2\%$ and $y = 1\%$.

Summary and Conclusions

In his article Debraj Ray (2014) criticizes Thomas Piketty:

"... And so we come to Piketty' s Third Fundamental Law, what he calls " the central contradiction of capitalism" : The rate of return on capital systematically exceeds the overall rate of growth of income: $r > g$.

Relatively speaking, this is the most interesting of the three laws. ...

Here is what Piketty concludes from this Law, as do several approving reviewers of his book: that because the rate of return on capital is higher than the rate of growth overall, the income of capital owners must come to dominate as a share of overall income. ...

(a) the above assertion is simply not true, or to be more precise, it may well be true but has little or nothing to do with whether or not $r > g$, and (b) the law itself is a simple consequence of a mild efficiency criterion that has been known for many decades in economics. Indeed, most economists know (a) and (b), or will see these on a little reflection.

. . .

But the Piketty faithful will still cling to the magic of that all-pervasive formula: $r > g$. That looks right, doesn' t it, and besides, it is impressive how empirically the law appears to hold through decades of data. My answer is: yes, it does look right, and its empirical validity is indeed impressive, but to me it is impressive for a different reason: that it is a mini-triumph of economic theory. Here is a fact. Take any theory of economic growth ... it follows, not empirically but as a matter of theoretical prediction, that $r > g$. Piketty' s Third Law has been known to economic theorists for at least 50 years, and no economic theorist has ever suggested that it "explains" rising inequality. Because it doesn' t. It can' t, because the models that generate this finding are fully compatible with stable inequalities of income and wealth.

. . .

You need something else to get at rising inequality. What then, explains the marked and disconcerting rise of inequality in the world today? Capital, in the physical and financial sense that Piketty uses it, has something to do with it. But it has something to do with it because it is a vehicle for accumulation. It is probably the principal vehicle for accumulation by the top 1% or the top 0.01%, simply because there are generally limits on how high the compensation to human capital can be in any generation. It is hard enough to make a few hundred thousand dollars in annual labor income, and reaching the million-dollar mark (let alone tens of millions) is far harder and riskier. But physical capital — land and financial assets — can be steadily and boundlessly accumulated. In this sense Piketty is right in turning the laser on capital. But, as I said, it' s just a vehicle.

Ray says, that "Piketty's Third Law has been known to economic theorists for at least 50 years, and no economic theorist has ever suggested that it "explains" rising inequality." The argument is, that $r > g$, i.e. $r > y$ with the notation used here, cannot explain rising inequality, because $r > y$ is a consequence of the standard models of growth theory, which are themselves "compatible with stable inequalities of income and wealth".

Is this the only possible conclusion that might be drawn? What if the standard models of neoclassical growth theory were not appropriate to explain the evolution of heterogeneous economies over time? What if the redistribution potential of the fact that capital income has to be financed by labor income of others were not reasonably taken into account? Ray acknowledges capital as a vehicle of boundless accumulation, but the fact that this accumulation has to be paid for by others, seems to be out of sight[25].

Anyway, the model presented here shows a clear correlation between $r > y$ and rising inequality. It supports the empirical findings of Piketty on a theoretical basis, and we draw the following conclusions:

1. In a constant parameter economy no polarization occurs, if:

(a) $r = 0$ and each household or no household saves;

(b) or if $y > r > 0$ and for each group $1 \leq i \leq N$

$$\frac{1}{y-r}\Delta_1^i + \frac{\alpha^i}{y}S_0 > 0. \tag{7.1}$$

2. Assume $\Delta_1^l < 0$ for some lower household group l. If $y > r > 0$ and if r approaches y then (7.1) will eventually be violated, and in this case polarization occurs although the growth rate exceeds the capital return rate.

3. If $\Delta_1^l < 0$ and if α^l is small enough for given $\beta^l > 0$, then again (7.1) will be violated, and polarization will occur even if $y > r > 0$; thus, the distribution of labor income also has a critical impact on polarization.

4. If $r \geq y$ polarization occurs eventually.

5. A cause for the redistribution of wealth is interest transfer. Capital income that is generated from profitable properties has to be financed by the labor income of other households.

6. If polarization occurs, then the differences in wealth of upper and lower household groups become snowballing; the redistribution is driven by a positive feedback mechanism.

7. In section 5.4 above we saw that interest transfer Z_t^i is negative for lower household groups and positive for upper household groups, irrespective of the relation between $r > 0$ and y. Thus, redistribution of wealth always occurs, but if economic growth is strong enough, then this might not attract special attention.

In a constant parameter economy, $y > r$ is necessary to prevent the economy from becoming polarized, and this is in accordance with the empirical analysis of Thomas Piketty (2014, 2015) which he carried out for many different countries.

Furthermore, the condition $y > r$ and its prospect of avoiding polarization might explain why

[25] For those who are not convinced of the power of this redistribution mechanism, I'd like to recommend playing the Monopoly game a few times. Even the unconditional basic income. Each time a player's token lands on or passes over GO, whether by throwing the dice or drawing a card, the Banker pays him/her a $200 salary" won't prevent the lower households" from becoming impoverished eventually while paying increasing interest transfer payments to the upper households" round by round.

there is an omnipresent demand for growth; the results of this paper and the simulations mentioned below suggest that if the economic rules of the game are not changed, eternal growth is the only way to prevent an economy from becoming polarized.

But unlimited growth is not possible in a finite world. Therefore, our results strongly indicate that unlimited capital income and very different labor income distributions lead to polarization and eventually to economic breakdown, as proposed and modeled by Hyman Minsky (1992) and Steve Keen (2010, 2011).

In this article, the influence of taxes is not considered. Taxes may have a strong impact on the results derived in this article as they usually cause a redistribution of wealth from the upper to the lower household groups. In Kremer (2013) it has been shown that dynamic analysis can be extended to allow for the state as an economic agent. While it is not possible in this case to derive closed formulas for the evolution of wealth of different household groups, simulations suggest that the conclusions of the current model remain essentially valid even when the prevailing fiscal system is taken into consideration. That is, the current system of taxation in Germany that favors capital income reduces the redistribution of wealth from the lower to the upper household groups, but it does not prevent the economy from becoming polarized.

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